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ON SUBHARMONIC SOLUTIONS OF HAMILTONIAN SYSTEMS.(U)

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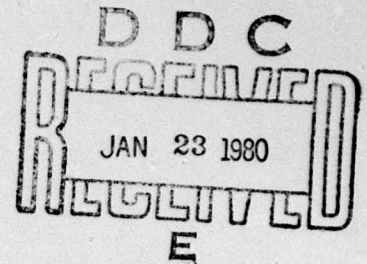
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UNIVERSITY OF WISCONSIN - MADISON  
MATHEMATICS RESEARCH CENTER

ON SUBHARMONIC SOLUTIONS OF HAMILTONIAN SYSTEMS

Paul H. Rabinowitz\*

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ABSTRACT

Let  $H: \mathbb{R} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$  and consider the corresponding Hamiltonian system of ordinary differential equations:  $(*) \dot{z} = J_z H_z(t, z)$  where  $H$  is  $T$  periodic in  $t$  and  $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ . Suppose further  $H$  is super (resp. sub) quadratic, i.e.  $H$  grows more (resp. less) rapidly than quadratically in  $z$  as  $|z| \rightarrow \infty$ . Using techniques from the calculus of variations, it is shown that under further hypotheses on  $H$ ,  $(*)$  possesses an infinite number of distinct solutions  $z_k$  having period  $kT$ .

AMS (MOS) Subject Classifications: 34C15, 58F05, 34C25, 58F05

Key Words: periodic solution; Hamiltonian system, subharmonics, critical point, variational methods, minimax.

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# SIGNIFICANCE AND EXPLANATION

Consider the Hamiltonian system of ordinary differential equations:

$$(*) \quad \frac{dp}{dt} = - \frac{\partial H}{\partial q} (t, p, q), \quad \frac{dq}{dt} = \frac{\partial H}{\partial p} (t, p, q) \quad \text{where } p = (p_1, \dots, p_n) \quad \text{and} \\ q = (q_1, \dots, q_n). \quad \text{Such equations model conservative forced mechanical systems.}$$

Suppose  $H$  is  $T$  periodic in  $t$ . Then one might hope for a  $T$  periodic response. Under appropriate conditions on  $H$ , it is shown that this is the case. Moreover  $(*)$  possesses a family of infinitely many distinct subharmonic solutions, i.e. solutions having period  $kT$  where  $k$  is an integer.

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Paul H. Rabinowitz\*

INTRODUCTION

Consider the Hamiltonian system of ordinary differential equations

$$(0.1) \quad \dot{z} = J H_z(t, z), \quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

where  $z \in \mathbb{R}^{2n}$  and  $H$  is  $T$  periodic in  $T$ . It is then natural to seek  $T$  periodic solutions of (0.1). Since  $H$  is  $kT$  periodic for all  $k \in \mathbb{N}$ , one can also search for  $kT$  periodic solutions (called subharmonics). This latter quest is complicated by the fact that any  $T$  periodic solution is a fortiori  $kT$  periodic. Thus an additional argument is required to show that any subharmonics are indeed distinct. Our main goal in this paper is to obtain the existence of subharmonic solutions for certain Hamiltonian systems which are either sub- or superquadratic, i.e. which grow either less or more rapidly than quadratically at  $\infty$  in an appropriate sense.

Some existence results for  $T$  periodic solutions of (0.1) were presented in (1) for superquadratic Hamiltonian systems using finite dimensional min-max arguments together with estimates suitable to pass to a limit. An improved existence mechanism was introduced in (2) and applied to some of the superquadratic problems of (1) as well as to several subquadratic cases. ~~We~~ <sup>It</sup> will show here that these problems possess not only one  $T$  periodic solutions  $(z_1)$   <sup>$z_{sub-1}$</sup>  but infinitely many distinct subharmonic solutions  $(z_k)$ . A word of caution must be entered at this point. Although  $(z_k)$  has period  $kT$ , it may not be the case that  $(z_k)$  has minimal (i.e. primitive) period  $kT$ . Indeed simple examples show that there may be an upper bound on the minimal period of  $(z_k)$ .

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Sections 1 and 2 are concerned with superquadratic Hamiltonian systems. In §1, using the variational framework of [2], we first establish the existence of a  $T$  periodic solution of (0.1) in a somewhat more general context than was treated in [1]. A comparison argument and some simple estimates then yield a family of distinct subharmonics. In §2 we study the behavior of  $z_k$  as  $k \rightarrow \infty$ . In particular under further hypotheses on  $H$  we show  $z_k \rightarrow 0$  and has a minimal period which tends to  $\infty$  as  $k \rightarrow \infty$ . Lastly similar results are obtained for a family of subquadratic Hamiltonian systems in §3.

We do not know of many works on subharmonic solutions of Hamiltonian systems in the global setting treated here. For a single second order equation, superquadratic results related to ours, but containing much more information, have been obtained by Jacobowitz [3] and by Hartman [4] using the Poincaré-Birkhoff fixed point theorem. See also Nehari [18] and Wolkowisky [19]. In work in progress, Clarke and Ekeland have shown there are a family of distinct subharmonics for a second order convex subquadratic Hamiltonian system [5].

Local results centered about the Birkhoff fixed point theorem and Birkhoff-Lewis Theorem have been the object of a considerable amount of study and establish the existence of long periodic solutions of (0.1) near an equilibrium or periodic solution. See e.g. Birkhoff [6] or Siegel-Moser [7, §24] for the Birkhoff fixed point theorem and applications to the restricted three body problem. More on such applications can be found in Moser [8] and Conley [9]. For the Birkhoff-Lewis Theorem, see e.g. Birkhoff [10], Birkhoff-Lewis [11], Lewis [12], Arnold [13], Harris [14], or Moser [15].

We thank Charles Conley and Jürgen Moser for several helpful conversations.

# §1 THE SUPERQUADRATIC CASE

We begin this section by proving the existence of one nontrivial solution to

$$(1.1) \quad \dot{z} = H_z(t, z).$$

Suppose  $H(t, z) = \mathcal{L}(z) + \hat{H}(t, z)$  with  $\mathcal{L}(z)$  a quadratic form and  $\hat{H}$  satisfies

$$(H_1) \quad \hat{H}(t, z) \geq 0, \quad t \in \mathbb{R}, \quad z \in \mathbb{R}^{2n}$$

$$(H_2) \quad \hat{H}(t, z) = o(|z|^2) \text{ at } z = 0$$

$$(H_3) \quad \text{There is a } \mu \in (2, \infty) \text{ and } R > 0 \text{ such that } 0 < \mu \hat{H}(t, z) \leq (z, \hat{H}_z(t, z))_{\mathbb{R}^{2n}} \\ \text{for all } t \in \mathbb{R}, \quad |z| \geq R$$

$$(H_4) \quad \text{There is a } T > 0 \text{ such that } \hat{H}(t + T, z) = \hat{H}(t, z) \text{ for all } t \in \mathbb{R}, \quad z \in \mathbb{R}^{2n}$$

$$(H_5) \quad \text{There are constants } \alpha, R_1 > 0 \text{ such that } |\hat{H}_z(t, z)| \leq \alpha(z, \tilde{H}_z(t, z))_{\mathbb{R}^{2n}} \\ \text{for all } t \in \mathbb{R}, \quad |z| > R_1.$$

In  $(H_3)$ ,  $(H_5)$ , and the sequel,  $(\cdot, \cdot)_{\mathbb{R}^Y}$  denotes the usual Euclidean inner product in  $\mathbb{R}^Y$ . Since  $\mathcal{L}$  is quadratic, there is a symmetric  $2n \times 2n$  matrix  $\odot$  such that  $\mathcal{L}(z) = \frac{1}{2}(\odot z, z)_{\mathbb{R}^{2n}}$ . Consider the eigenvalue problem

$$(1.2) \quad \odot \xi = \lambda \xi$$

Since  $\mathcal{L}$  is real, whenever  $(\lambda, \xi)$  is an eigenpair for (1.2), so is  $(\bar{\lambda}, \bar{\xi})$ . Suppose

(2) (1.2) has  $2n$  purely imaginary eigenvalues  $\lambda_j = i\mu_j$  where  $\mu_j > 0$ ,  $1 \leq j \leq n$  and  $\lambda_{j+n} = \bar{\lambda}_j$ ,  $1 \leq j \leq n$  with corresponding eigenvectors  $\xi_j$ ,  $1 \leq j \leq n$  and  $\xi_{j+n} = \bar{\xi}_j$ ,  $1 \leq j \leq n$ .

Note that  $\mathcal{L}$  need not be definite. We can normalize the  $\xi_j$  so that

$$(1.3) \quad |(\ominus \xi_j, \xi_h)_{\mathbb{R}^{2n}}| = \frac{2}{\pi} \delta_{jk}$$

and  $\delta_{jk}$  is the usual Kronecker  $\delta$ .

Theorem 1.4: Suppose  $H \in C^1(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$  and satisfies  $(H_1) - (H_5)$  and  $(\mathcal{L})$ . Then (1.1) possesses a nonzero  $T$  periodic solution.

Corollary 1.5: Suppose  $H \in C^1(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$  and satisfies  $(H_1) - (H_5)$  and  $\ominus = 0$ . Then (1.1) possesses a nonconstant solution.

Remark 1.6: If  $H$  is independent of  $t$ , these results reduce to Theorem 3.3 and Corollary 3.54 of [2] and are true for all  $T > 0$  without assuming  $(H_5)$ . Indeed still stronger statements are valid for the autonomous case [16]. The proofs of Theorem 1.4 and Corollary 1.5 are similar to the proofs given in [2]. Since the dependence of certain sets on parameters is essential in getting the subharmonic solutions of (1.1) later, we will sketch the proof of Theorem 1.4 carrying out in detail those parts of the argument which differ from [2] and where precise estimates are necessary in the sequel. We will not prove Corollary 1.5. Its proof is similar to but simpler than that of Theorem 1.4.

The existence lemma which Theorem 1.4 requires was proved in [1]:

Lemma 1.7: Suppose  $E$  is a real Hilbert space,  $E = E_1 \oplus E_2$  where  $E_2 = E_1^\perp$  and  $f \in C^1(E, \mathbb{R})$  satisfies:

$$(f_1) \quad f(u) = \frac{1}{2} (Lu, u) + b(u) \quad \text{where } u = u_1 + u_2 \in E_1 \oplus E_2, \quad Lu = L_1 u_1 + L_2 u_2,$$

and  $L_i: E_i \rightarrow E_i$  is bounded linear and self adjoint,  $i = 1, 2$ .

(f<sub>2</sub>)  $b(u)$  is weakly continuous and is uniformly differentiable on bounded sets

(f<sub>3</sub>) All sequences  $(u_m)$  such that  $f(u_m)$  is bounded from above and  $f'(u_m) \rightarrow 0$  are bounded

(f<sub>4</sub>) There are Hilbert manifolds  $S$  and  $Q$ ,  $Q$  with boundary,  $v \in E_2$ , and  $\alpha > \omega$  such that

- (i)  $SC\{v\} + E_1$  and  $f \geq \alpha$  on  $S$ ,
- (ii)  $f \leq \omega$  on  $\partial Q$ ,
- (iii)  $S$  and  $\partial Q$  link

Then  $f$  possesses a critical value  $c \geq \alpha$ .

Remark 1.8: The precise definition of linking is given in [2]. We will omit it here but suffice it to say that (f<sub>4</sub>) (iii) is satisfied for superquadratic nonlinearities  $b(u)$  if  $v = 0$ ,  $S = \partial B_\rho \cap E_1$ , and  $Q = \{se_1 \mid 0 \leq s \leq r_1\} \oplus (B_{r_2} \cap E_2)$  where  $B_r$  denotes a ball of radius  $r$  in  $E_1$ ,  $r_1 > \rho > 0$ ,  $r_2 > 0$ , and  $e_1 \in \partial B_1 \cap E_1$ . Likewise in §3 we will use the fact that (f<sub>4</sub>) (iii) is satisfied for subquadratic nonlinearities if  $S = \{v\} + E_1$  and  $Q = B_R \cap E_2$  where  $R > \|v\|$ .

Remark 1.9: The critical value  $c$  of Lemma 1.7 can be characterized as the minimum of  $f$  over an appropriate class of sets [2]. For our later purposes it suffices to observe that  $Q$  is one of these sets and therefore

$$(1.10) \quad c \leq \sup_{u \in Q} f(u)$$

Moreover a sharper lower bound for  $c$  is

$$c \geq \inf_{u \in S} f(u)$$

Proof of Theorem 1.4: Before verifying the hypotheses of Lemma 1.7, it is

necessary because of the unrestricted rate of growth for  $\hat{H}$  at  $\infty$  to carry out a technical modification of the problem. Let  $K > 0$ ,  $\chi(s) = 1$  for  $s \leq K$ ;  $\chi(s) = 0$  for  $s \geq K + 1$ , and  $\frac{d\chi}{ds} < 0$  for  $s \in (K, K + 1)$  with  $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ . Set

$$(1.11) \quad H_K(t, z) = \mathbf{z}(z) + \chi(|z|) \hat{H}(t, z) + (1 - \chi(|z|)) r |z|^4.$$

For  $r$  (depending on  $K$ ) sufficiently large,  $H_K$  satisfies  $(H_1) - (H_5)$  with  $\alpha$  replaced by  $\bar{\alpha} = \alpha + 2$  and  $\mu$  replaced by  $\bar{\mu} = \min(\mu, 4)$ . Therefore to prove the theorem, it suffices to find a nontrivial solution of

$$(1.12) \quad \dot{z} = \int_{Kz} H_K(t, z)$$

with  $\|z\|_{L^\infty} \leq K$ . This we will carry out next.

To put (1.12) into the framework of Lemma 1.7, without loss of generality we can take  $T = 2\pi$ . Let  $E = (W^{\frac{1}{2}, 2}(S^1)^{2n})$ , the Hilbert space of  $2n$  tuples of  $2\pi$  periodic functions which possess  $\frac{1}{2}$  of a derivative which is square integrable. More precisely the usual norm for  $E$  is

$$\|z\|^2 = \sum_{j \in \mathbb{Z}} (1 + |j|) |a_j|^2$$

where  $z = \sum a_j e^{ijt}$ . However we shall introduce an equivalent norm below which is more suitable for our estimates.

Let  $\phi_{jm} = \frac{1}{2i} [e^{imt} \xi_j - e^{-imt} \bar{\xi}_j]$ ,  $1 \leq j \leq 2n$ ,  $m \in \mathbb{Z}$ . It is easy to see that these functions span  $E$ . Moreover  $\phi_{jm}$  satisfies

$$(1.13) \quad j\dot{z} = \frac{m}{\mu_j} \odot z$$

Setting

$$A(z) = \int_0^{2\pi} (p, q)_{\mathbb{R}^n} dt$$

for smooth  $z$ , where  $z = (p, q)$ , then

$$\hat{A}(z) \equiv A(z) - \int_0^{2\pi} \mathcal{L}(z) dt = \left( \frac{m}{u_j} - 1 \right) \operatorname{sgn} (\odot \xi_j, \xi_j)_{\mathbb{R}^{2n}} \equiv \sigma_{jm}$$

for  $z = \varphi_{jm}$  via (1.3) where for  $s \neq 0$ ,  $\operatorname{sgn} s = 1$  if  $s > 0$  and equals  $-1$  if  $s < 0$ . Now set  $E^+ = \operatorname{span} \{\varphi_{jm} | \sigma_{jm} > 0\}$ ,  $E^- = \operatorname{span} \{\varphi_{jm} | \sigma_{jm} < 0\}$  and  $E^0 = \operatorname{span} \{\varphi_{jm} | \sigma_{jm} = 0\}$ . Note that  $E^0$  has dimension  $\leq 2n$  and may be empty. These spaces are complementary subspaces of  $E$ . Therefore  $z \in E$  implies  $z = z^+ + z^- + z^0 \in E^+ \oplus E^- \oplus E^0$ . We can and will take as norm in  $E$

$$\|z\|^2 = \hat{A}(z^+) - \hat{A}(z^-) + \int_0^{2\pi} |\mathcal{L}|(z^0) dt$$

where for  $z = \varphi_{jm}$ ,  $|\mathcal{L}|(z) = \frac{2}{\pi}$  and is defined elsewhere by bilinearity. It then follows that  $E^+$ ,  $E^0$  are orthogonal subspaces of  $E$ . Note also that  $E$  is compactly embedded in  $L^s$  for all  $s \in (1, \infty)$ , i.e.

$$(1.14) \quad \|z\|_{L^s} \leq a_s \|z\|$$

for all  $z \in E$  [16].

Now set  $E_1 = E^+$ ,  $E_2 = E^0 \oplus E^-$  and

$$(1.15) \quad f(z) = A(z) - \int_0^{2\pi} H_K(t, z) dt$$

It then follows as in [2] from  $(H_2)$  and the form of  $H_K$  that  $f \in C^1(E, \mathbb{R})$  and satisfies  $(f_1) - (f_2)$  where  $\frac{1}{2}(Lz, z) = \hat{A}(z)$ . To verify  $(f_3)$ , we modify Lemma 3.25 of [1]. Note first that  $(H_3)$  implies there are constants  $a_1, a_2 \geq 0$  such that

$$(1.16) \quad \hat{H}_K(t, z) \equiv H_K(t, z) - \mathcal{L}(z) \geq a_1 |z|^{\bar{u}} - a_2$$

for all  $z \in \mathbb{R}^{2n}$  where  $a_1, a_2$  are independent of  $K$ . In (1.16) and the

remainder of this paper  $a_i, M_j$  repeatedly denote nonnegative constants. Suppose that  $(z_m)$  is a sequence such that  $f(z_m) \leq M$  and  $f'(z_m) \rightarrow 0$  as  $m \rightarrow \infty$ . Then for large  $m$  with  $z = z_m$  we have

$$(1.17) \quad M + \frac{1}{2} \|z\| \geq f(z) - \frac{1}{2} f'(z)z = \int_0^{2\pi} \left[ \frac{1}{2} (z, \hat{H}_{Kz}(t, z))_{\mathbb{R}^{2n}} - \hat{H}_K(t, z) \right] dt$$

$$\geq \left( \frac{1}{2} - \frac{1}{\mu} \right) \int_0^{2\pi} (z, H_{Kz})_{\mathbb{R}^{2n}} dt - M_1$$

via  $(H_3)$  for  $\hat{H}_K$ . The constant  $M_1$  is independent of  $K$ . The form of  $\hat{H}_K$  and (1.17) imply

$$(1.18) \quad \|z\|_L^4 \leq M_2 + M_3 \|z\|$$

with  $K$  dependent  $M_1, M_2$ . Moreover by (1.17), (1.16), and the Hölder inequality,

$$(1.19) \quad 1 + \|z\| \geq M_4 \|z\|_{L^\mu}^{\frac{\mu}{\mu-1}} \geq M_5 \|z\|_{L^2}^{\frac{\mu}{\mu-1}}.$$

Since on  $(L^2(S^1))^{2n} \left\{ \int_0^{2\pi} |z|(z) dt \right\}^{\frac{1}{2}}$  is equivalent to  $\|\cdot\|_{L^2}$ , (1.19) yields

$$(1.20) \quad 1 + \|z\| \geq M_6 \left( \int_0^{2\pi} |z|(z) dt \right)^{\mu/2}.$$

Writing  $z = z^+ + z^- + z^0$  and using the orthogonality of these functions with respect to the " $z$ " norm, we find

$$(1.21) \quad M_7 (1 + \|z\|) \geq \|z^0\|_{L^\mu}^{\frac{\mu}{\mu-1}}.$$

Setting  $\zeta = z^+$  in

$$(1.22) \quad |f'(z) \zeta| \leq \|\zeta\|$$

yields

$$(1.23) \quad 2 \|z^+\|^2 = 2 \hat{A}(z^+) \leq \|z^+\| + \left| \int_0^{2\pi} (z^+, \hat{H}_{Kz}(t, z^+))_{\mathbb{R}^{2n}} dt \right|$$

$$\leq \|z^+\| + M_8 \left( 1 + \|z\|_{L^4}^3 \right) \|z^+\|_{L^4}$$

or

$$(1.24) \quad \|z^+\| \leq M_9 (1 + \|z\|)^{3/4}$$

via (1.14). Combining (1.24) with a similar inequality for  $z^-$  and (1.21) gives

$$(1.25) \quad \|z\| \leq M_{10} \left( 1 + \|z\|^{3/4} + \|z\|^{\bar{\mu}} \right)$$

whence follows the boundedness of  $(z_m)$  and  $(f_3)$ .

To verify  $(f_4)$ , the construction of  $S = \partial B_\rho \cap E_1$  is essentially as in [2]. To obtain  $Q$  with  $r_1$  and  $r_2$  independent of  $K$ , let  $e \in \partial B_1 \cap \hat{E}_1$  and  $z = z^0 + z^- \in \hat{E}_2$ . Then

$$(1.26) \quad f(z + se) = s^2 - \|z^-\|^2 - \int_0^{2\pi} \hat{H}_K(t, z + se) dt$$

By  $(H_1)$ ,  $f(z) \leq 0$ . Moreover as in (1.19) - (1.21),

$$(1.27) \quad \int_0^{2\pi} H_K(t, z + se) dt \geq a_3 (\|z^0\|^{\bar{\mu}} + s^{\bar{\mu}}) - a_4$$

holds with  $a_3, a_4$  independent of  $K$ . Thus by (1.26) - (1.27)

$$(1.28) \quad f(z + se) \leq s^2 - \|z^-\|^2 - a_3 (\|z^0\|^{\bar{\mu}} + s^{\bar{\mu}}) + a_4.$$

Choose  $r_1$  so that

$$(1.29) \quad \psi(s) \equiv s^2 - a_3 s^{\bar{\mu}} + a_4 \leq 0$$

for all  $s \geq r_1$ . Set  $M = \max_{s \in [0, r_1]} \psi(s)$  and observe that  $\|z^-\|^2 + a_3 \|z^0\|^{\bar{\mu}} \rightarrow \infty$  as  $\|z\|^{\bar{\mu}} \rightarrow \infty$ . Hence for  $z \in \partial B_{r_2} \cap E_2$  and  $r_2$  sufficiently large, we have

$$M \leq \|z^-\|^2 + a_3 \|z^0\|^{\bar{\mu}} \text{ and } f \leq 0 \equiv \omega \text{ on } \partial Q \text{ with } Q = \{s \mid 0 \leq s \leq r_1\} \oplus (B_{r_2} \cap E_2).$$

Since  $f$  satisfies the hypotheses of Lemma 1.7, it possesses a positive critical value  $c$  and corresponding critical point  $z_K$ . It follows from [2] that  $z_K$  is a classical solution of (1.12) and  $z_K$  is nonzero since  $f(z_K) > 0$ . It remains only to prove that  $z_K$  satisfies (1.1) for appropriately chosen  $K$ . By Remark 1.9,

$$(1.30) \quad c \leq \sup_{z \in Q} f(z) = \sup_{\|z^0 + z^-\| \leq r_2, r \in [0, r_1]} s^2 - \|z^-\|^2 - \int_0^{2\pi} H_K(t, z) dt \leq r_1^2$$

Now as in (1.17), (1.19),

$$(1.13) \quad M_1 + r_1^2 \geq \left( \frac{1}{2} - \frac{1}{\mu} \right) \left| \int_0^{2\pi} (z_K, \hat{H}_{Kz})_{\mathbb{R}^{2n}} dt \right| \geq a_5 \|z_K\|_{L^{\bar{\mu}}}^{\bar{\mu}} - a_6$$

Hence

$$(1.32) \quad \|z_K\|_{L^{\bar{\mu}}} \leq M_2 \text{ (independently of } K)$$

By  $(H_5)$  for  $H_K$  and (1.30) - (1.32),

$$(1.33) \quad \|\dot{z}_K\|_{L^1} \leq \|g_{H_{Kz}}(t, z_K)\|_{L^1} \leq \|\odot z_K\|_{L^1} + \|\hat{H}_{Kz}(t, z_K)\|_{L^1} \leq M_3.$$

Finally by (1.33) and (1.32),

$$(1.34) \quad \|z_K\|_{L^\infty} \leq \|\dot{z}_K\|_{L^1} + a_7 \|z_K\|_{L^p} \leq M_4.$$

Thus for  $K \geq M_4$  we have  $H_{Kz}(t, z_K) = H_z(t, z_K)$ , (1.1) is satisfied, and the proof is complete.

Remark 1.35: If  $H$  is independent of  $t$  and  $\odot$  is not positive definite, (1.1) possesses at least one nonzero equilibrium solution. This may be the solution obtained in Theorem 1.4. A deeper result in [16] gives nonconstant solutions for this case assuming only  $(H_3)$ . However if  $\odot$  is positive definite or  $\odot = 0$ ,  $f(z_K) > 0$  and  $H(z) \geq 0$  imply  $z_K$  is nonconstant.

Having established the existence of one nonzero solution of (1.1), we will show that in fact there are infinitely many.

Theorem 1.36: Under the hypotheses of Theorem 1.4, there exists a sequence  $(k_j) \subset \mathbb{N}$ ,  $k_j \rightarrow \infty$ , and corresponding distinct  $k_j T$  periodic solutions of (1.1).

Proof: We can take  $T = 2\pi$ . Choose  $k \in \mathbb{N}$ . It is convenient to make the change of variables  $\tau = k^{-1}t$ . Thus if  $z(t)$  is a  $2\pi k$  periodic solution of (1.1),  $\zeta(\tau) = z(k\tau)$  satisfies

$$(1.37) \quad \frac{d\zeta}{d\tau} = k g_{H_z}(k\tau, \zeta)$$

and we seek a  $2\pi$  periodic solution of (1.37). Since  $k H(k\tau, z)$  satisfies  $(H_1) - (H_5)$  and (2), Theorem 1.4 provides a critical point  $\zeta_k(\tau) \in E$  of

$$(1.38) \quad f_k(\zeta) = A(\zeta) - k \int_0^{2\pi} H_k(k\tau, \zeta) d\tau$$

where  $K$  depends on  $k$ , which for appropriately large  $K$  is a classical solution of (1.37). Note that  $\zeta_1(k\tau)$  also satisfies (1.37). If  $\zeta_1(k\tau) = \zeta_k(\tau)$ , it is easy to check that

$$(1.39) \quad c_k \equiv f_k(\zeta_k) = k f_1(\zeta_1) = k c_1$$

Since  $c_1 > 0$  by Lemma 1.7, it follows that  $c_k \rightarrow \infty$  as  $k \rightarrow \infty$ . We will show next that this is impossible since  $c_k$  is bounded from above independently of  $k$ .

Recall from (1.30) that

$$(1.40) \quad c_k \leq r_1^2(k)$$

where we have written  $r_1(k)$  to emphasize its dependence on  $k$ . The parameter  $r_1$  was determined in (1.29) which in turn was derived from (1.26) - (1.28).

The corresponding equation satisfied by  $r_1(k)$  is

$$(1.41) \quad \psi_k(s) = s^2 - k a_3 s^{\bar{\mu}} + k a_4 \leq 0$$

for all  $s \geq r_1(k)$ . It follows that

$$(1.42) \quad r_1(k) \leq \max \left\{ \left( \frac{2}{k a_3} \right)^{\frac{1}{\bar{\mu}-2}}, \left( \frac{2 a_4}{a_3} \right)^{\frac{1}{\bar{\mu}}} \right\}.$$

Thus the critical values  $c_k$  are uniformly bounded and therefore there is a  $k_1 \in \mathbb{N}$  such that  $\zeta_k(\tau) \neq \zeta_1(k\tau)$  for all  $k \geq k_1$ . Reapplying what we have just shown to the  $2\pi$  periodic function  $k_1 H(k_1 \tau, z)$ , it follows that there is a sequence of nonzero  $2\pi$  periodic solutions  $z_j(\tau)$  to

$$(1.43) \quad \frac{dz}{d\tau} = j k_1 \phi_H(j k_1 \tau, z)$$

with  $z_j(\tau) \neq z_1(j\tau)$  for all  $j \geq k_2$ . Moreover from the form of (1.43) and

the corresponding variational problem,  $z_j(\tau) = \zeta_{jk_1}(\tau)$  and  $z_j(\tau) \neq \zeta_1(jk, \tau)$  for all  $j \geq k_1$ . It follows that we have a sequence  $\zeta_1(t), \zeta_{k_1}(t/k_1), \zeta_{k_1 k_2}(t/k_1 k_2), \dots$ , of distinct nonzero solutions of (1.1) and the proof is complete.

Remark 1.44: The argument of the last paragraph can be replaced by the following one which gives a wider range of periods for the solutions so obtained. If for some  $k > m$ ,  $\zeta_k(\tau) = \zeta_k(t/k) \equiv \zeta(t) = \zeta_m(t/m)$ , then  $\zeta(k\tau) = \zeta_k(\tau)$  and  $\zeta(m\tau) = \zeta_m(\tau)$ . Moreover as in (1.39),

$$(1.45) \quad c_k = f_k(\zeta_k) = kf_1(\zeta); \quad c_m = f_m(\zeta_m) = mf_1(\zeta)$$

Since  $f_1(\zeta) > 0$  and  $\{c_j\}$  is bounded, it follows that there can be at most finitely many  $k > m$  such that  $\zeta_k = \zeta_m$ .

Remark 1.46: If  $H$  is independent of  $t$  and  $z(t)$  is a solution of (1.1) so is  $z_\theta(t) = z(t + \theta)$  for all  $\theta \in [0, 2\pi)$ . Since  $f_k(z_\theta) = f_k(z)$ , the solutions obtained above must differ by more than a simple translation for the case.

Remark 1.47: If  $H(t, z)$  splits into the sum of kinetic and potential energy terms, e.g.  $H(t, z) = \frac{1}{2} |p|^2 + V(t, q)$ , simpler arguments without requiring  $(H_5)$  can be used to establish analogues of results of this and the following section. (See also [1])

## §2 ASYMPTOTICS

In this section we will study when the functions  $\zeta_k$  obtained in Theorem 1.36 are uniformly bounded in  $\|\cdot\|_{C^1}$  and therefore possess a limit, when this limit is zero, and when the minimal period of  $\zeta_k$  tends to infinity as  $k \rightarrow \infty$ . For convenience we always assume  $T = 2\pi$ . Let  $2\pi\ell_k^{-1}$  denote the minimal period of  $\zeta_k(\tau)$ . Then  $\zeta_k(t/k)$  has minimal period  $2\pi k\ell_k^{-1}$ .

**Proposition 2.1:** Under the hypotheses of Theorem 1.4, if  $k\ell_k^{-1} \not\rightarrow \infty$  along some subsequence, then the functions  $\zeta_k(t/k)$  are uniformly bounded in  $\|\cdot\|_{C^1}$ .

**Proof:** Since

$$\begin{aligned} (2.2) \quad c_k &= A(\zeta_k) - k \int_0^{2\pi} H(k\tau, \zeta_k) d\tau \\ &= k \int_0^{2\pi} \left[ \frac{1}{2}(\zeta_k(\tau), \hat{H}_z(k\tau, \zeta_k))_{\mathbb{R}^{2n}} - \hat{H}(k\tau, \zeta_k) \right] d\tau \\ &= \ell_k \int_0^{\frac{2\pi k}{\ell_k}} \left[ \frac{1}{2}(\zeta_k(\frac{t}{k}), \hat{H}_z(t, \zeta_k))_{\mathbb{R}^{2n}} - \hat{H}(t, \zeta_k) \right] dt, \end{aligned}$$

by  $(H_3)$ ,

$$(2.3) \quad \frac{c_k}{\ell_k} \geq a_1 \int_0^{\frac{2\pi k}{\ell_k}} (\zeta_k, \hat{H}_z(t, \zeta_k))_{\mathbb{R}^{2n}} dt - a_2 \frac{2\pi k}{\ell_k}$$

Therefore

$$(2.4) \quad \frac{c_k}{\ell_k} + a_3 \geq a_1 \int_0^{\frac{2\pi k}{\ell_k}} (\zeta_k, \hat{H}_z(t, \zeta_k))_{\mathbb{R}^{2n}} dt \geq a_4 \int_0^{\frac{2\pi k}{\ell_k}} |\zeta_k|^\mu dt$$

via (1.14). The Hölder inequality then gives a bound on  $\|\zeta_k\|_{L^1}$ . If

$|\zeta_k(\frac{t}{k})| \geq R_1$  in the interval  $(s, \sigma)$ , then for  $t \in (s, \sigma)$ ,

$$\begin{aligned}
(2.5) \quad \left| \zeta_k \left( \frac{t}{k} \right) \right| &\leq \left| \zeta_k \left( \frac{s}{k} \right) \right| + \int_s^0 |H_z(x, \zeta_k)| dx \\
&\leq \left| \zeta_k \left( \frac{s}{k} \right) \right| + \int_0^{2\pi k/\ell_k} (|\ominus \zeta_k| + |\hat{H}_z(x, \zeta_k)|) dx \\
&\leq \left| \zeta_k \left( \frac{s}{k} \right) \right| + M_1
\end{aligned}$$

by (2.3) and  $(H_5)$  and the  $L^1$  bound. Integrating (2.5) over  $[0, 2\pi k\ell^{-1}]$  with respect to  $s$ , our  $L^1$  bound implies that the functions are uniformly bounded in  $L^\infty$  and by (1.1) in  $C^1$ .

Remark 2.6: It follows immediately from Proposition 2.1 that a subsequence of the functions  $\zeta_k \left( \frac{t}{k} \right)$  converge in  $C^1$  to a periodic solution of (1.1)

If  $H$  is independent of  $t$ , we can improve on Proposition 2.1

Proposition 2.7: If  $H = H(z)$  satisfies  $(H_1) - (H_3)$ , the functions  $\zeta_k$  are uniformly bounded in  $C^1$ .

Proof: As in (2.2),

$$\begin{aligned}
(2.8) \quad c_k &= \int_0^{2\pi k} \left[ \frac{1}{2} (\zeta_k, \hat{H}_z(\zeta_k))_{\mathbb{R}^{2n}} - \hat{H}(\zeta_k) \right] dt \\
&\geq a_5 \int_0^{2\pi k} |(\zeta_k, \hat{H}_z(\zeta_k))_{\mathbb{R}^{2n}}| dt - ka_6
\end{aligned}$$

Since  $H(\zeta_k) \equiv \text{constant}$ ,

$$\begin{aligned}
(2.9) \quad 2\pi k H(\zeta_k) &= \int_0^{2\pi k} H(\zeta_k) dt \leq \mu^{-1} \int_0^{2\pi k} |(\zeta_k, \hat{H}_z(\zeta_k))_{\mathbb{R}^{2n}}| dt \\
&\quad + a_7 k \|\zeta_k\|_{L^\infty}
\end{aligned}$$

Now (1.16) and (2.8) yield an  $L^\infty$  bound for  $\zeta_k$  and (1.1) provides the  $C^1$  bound.

Next we study a situation in which  $\zeta_k \rightarrow 0$  as  $k \rightarrow \infty$ . For this we require a global version of  $(H_3)$ :

$(\hat{H}_3)$ : There is a  $\mu \in (2, \infty)$  such that  $0 < \mu \hat{H}(t, z) \leq (z, \hat{H}_z(t, z))_{\mathbb{R}^{2n}}$   
for all  $z \in \mathbb{R}^{2n} \setminus \{0\}$  and  $t \in \mathbb{R}$

Obviously  $(\hat{H}_3)$  implies  $(H_3)$  and in addition for the corresponding  $\hat{H}_K$ , for all  $r \geq \rho > 0$  and  $\zeta \in S^{2n-1}$ ,

$$(2.10) \quad \hat{H}_K(t, r\zeta) > \hat{H}_K(t, \rho\zeta) \left(\frac{r}{\rho}\right)^\mu$$

Choosing in particular  $r = 1$ , we see  $(\hat{H}_3)$  also implies  $(H_2)$ .

Theorem 2.11: Suppose  $H \in C^1(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$  and satisfies  $(H_1)$ ,  $(\hat{H}_3)$ ,  $(H_4)$  -  $(H_5)$  and (2). Then (0.1) possesses a family of  $kT$  periodic solutions which converge to 0 as  $k \rightarrow \infty$  uniformly in  $C^1(\mathbb{R}, \mathbb{R}^{2n})$ .

Proof: By our preliminary remarks, Theorem 1.36 is applicable here and we have a family of solutions  $\zeta_k(t/k)$  as in that result. We will obtain uniform  $L^\infty$  and  $C^1$  bounds for these functions. By (2.2) and  $(\hat{H}_3)$ ,

$$(2.12) \quad \begin{aligned} c_k &\geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_0^{2\pi k} (\zeta_k, \hat{H}_z(t, \zeta_k))_{\mathbb{R}^{2n}} dt \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \mu \int_0^{2\pi k} \hat{H}(t, \zeta_k) dt \end{aligned}$$

Let  $I \subset [0, 2\pi k]$  be an interval in which  $|\zeta_k(\frac{t}{k})|^\mu \geq R_1$ . We can assume  $R_1 > 1$ .

If  $I = [0, 2\pi k]$ , then by (2.10)

$$(2.13) \quad \int_0^{2\pi k} H(t, \zeta_k) dt \geq \gamma \int_0^{2\pi k} |\zeta_k|^\mu dt \geq 2\pi k \gamma R_1^\mu \rightarrow \infty$$

as  $k \rightarrow \infty$  where  $\gamma = \min_{t \in \mathbb{R}, |\zeta|=1} H(t, \zeta)$ . But since  $c_k$  is uniformly bounded in  $k$ , (2.13) is contrary to (2.12). Hence for all large  $k$ , by lengthening  $I$  if necessary we can assume there is an  $s \in \partial I$  such that  $|\zeta_k(\frac{s}{k})| = R_1$ . Now for any  $t \in I_k$ , as in (2.5),

$$(2.14) \quad \left| \zeta_k\left(\frac{t}{k}\right) \right| \leq \left| \zeta_k\left(\frac{s}{k}\right) \right| + \int_s^t (|\ominus \zeta_k| + |\hat{H}_z(x, \zeta_k)|) dx \\ \leq R_1 + a_1 \int_s^t |\zeta_k| dx + a \int_s^t (\zeta_k, \hat{H}_z(x, \zeta_k))_{\mathbb{R}^{2n}} dx$$

Since on  $I$ ,

$$(2.15) \quad (\zeta_k, \hat{H}_z(t, \zeta_k))_{\mathbb{R}^{2n}} \geq \mu \hat{H}(t, \zeta_k) \geq \mu \gamma |\zeta_k|^\mu \geq \mu \gamma |\zeta_k|,$$

we can estimate the second term on the right in (2.14) in terms of the third term. Then (2.14) and (2.12) give an  $L^\infty$  bound and (1.1) a  $C^1$  bound for  $\zeta_k$ . By these bounds,  $\zeta_k$  converge along some subsequence uniformly on compact subsets of  $\mathbb{R}$  to  $Z$ . For  $m \geq k$  by (2.12) and  $(\hat{H}_3)$ ,

$$c_m \geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_0^{2\pi k} (\zeta_m, \hat{H}_z(t, \zeta_m))_{\mathbb{R}^{2n}} dt.$$

Letting  $m \rightarrow \infty$  gives

$$\overline{\lim} c_m \geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_0^{2\pi k} (Z, \hat{H}_z(t, Z))_{\mathbb{R}^{2n}} dt$$

for all  $k \in \mathbb{N}$ . Hence

$$(2.16) \quad \overline{\lim} c_m \geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_0^\infty (Z, \hat{H}_z(t, Z))_{\mathbb{R}^{2n}} dt > 0$$

unless  $Z \equiv 0$ .

We will use a comparison agreement to show that  $c_m \rightarrow 0$  and therefore  $z \equiv 0$ . For any  $\sigma > 0$ , by (2.10),

$$(2.17) \quad \hat{H}_K(t, z) \geq b(\sigma) \left( \frac{|z|}{\sigma} \right)^{\bar{\mu}}$$

for  $|z| \geq \sigma$  where  $b(\sigma) = \min_{t \in \mathbb{R}, |\zeta| = \sigma} H_K(t, \zeta)$ . Let  $\chi_\omega \in C^\infty(\mathbb{R}, \mathbb{R})$  satisfy  $\chi_\omega(s) = 1$  for  $s \leq \omega$ ;  $= 0$  for  $s \geq 2\omega$ ; and  $\chi'(s) < 0$  for  $s \in (\omega, 2\omega)$ . Choosing  $\sigma \in (0, \frac{1}{2})$ , define

$$(2.18) \quad G(s) = \beta[(1 - \chi_\sigma(s))b(\sigma) \left(\frac{s}{\sigma}\right)^{\bar{\mu}}] \chi_1(s) + (1 - \chi_1(s)) \gamma s^{\bar{\mu}}$$

Then for  $\beta$  sufficiently small,

$$(2.19) \quad G(|z|) \leq \hat{H}_K(t, z)$$

for all  $t \in \mathbb{R}$ ,  $z \in \mathbb{R}^{2n}$ . Moreover it is easy to verify that for  $\beta$  possibly still smaller,

$$(2.20) \quad 0 < \bar{\mu} G \leq (z, G_z)_{\mathbb{R}^{2n}}$$

for  $|z| \geq 2\sigma$  and  $\beta$  is independent of  $\sigma \in (0, \frac{1}{2})$ . Since  $G \equiv 0$  for  $|z| \leq \sigma$ , it follows that  $G$  satisfies  $(H_1) - (H_3)$  and is independent of  $t$ . Set

$$(2.21) \quad g_k(z) = A(z) - k \int_0^{2\pi} (\mathcal{L}(z) + G(|z|)) d\tau$$

It then follows from Remark 1.6 that for each  $k \in \mathbb{N}$ ,  $g_k$  has a critical value  $d_k$  and corresponding critical point  $w_k$ . By (2.19),

$$(2.22) \quad f_k(z) \leq g_k(z)$$

for all  $z \in E$ . The construction of  $S$  and  $Q$  (which depend on  $k$ ) given in Theorem 1.4 and [2] shows the same  $S$  and  $Q$  can be used in the

variational problem which determines  $c_k$  and  $d_k$ . Moreover by (2.22) any set which is admissible for the minimax characterization of  $d_k$  is also admissible for  $c_k$  (See p. 247-248 in [2]). Therefore by (2.22),

$$(2.23) \quad 0 < c_k \leq d_k.$$

We will show  $d_k \rightarrow 0$  as  $k \rightarrow \infty$ .

The proof of Theorem 1.4 with  $\hat{H}$  replaced by  $k\hat{H}$  shows that  $Q \subset \text{span} \{e_k\} \oplus \hat{E}_2^k \equiv F_k$  where now  $e_k \in E_1^k \cap \partial B_1$ ,

$$E_1^k = \text{span} \left\{ \varphi_{jm} \left| \left( \frac{m}{\mu_j} - k \right) \text{sgn} \left( \Theta_{\xi_j, \xi_j}^{2n} \right) > 0 \right. \right\},$$

etc. Hence by Remark 1.9,

$$(2.24) \quad 0 < d_k \leq \sup_{F_k} g_k$$

For  $w \in F_k$ , we have  $w = \|w\|_{L^2} \zeta$  where  $\zeta = \beta e_k + \delta v$  with  $v \in \hat{E}_2^k \cap \partial B_1$  and

$$(2.25) \quad \|\zeta\|_{L^2} = 1 \geq a_2 \left( \int_0^{2\pi} |\mathcal{L}|(\zeta) d\tau \right)^{\frac{1}{2}} = a_2 \left( \int_0^{2\pi} (\beta^2 |\mathcal{L}|(e_k) + \delta^2 |\mathcal{L}|(v)) d\tau \right)^{\frac{1}{2}} \geq a_3 \beta^2$$

with  $a_3$  independent of  $k$ . Choose any  $w \in F_k$  such that  $g_k(w) > 0$ . Then by (2.25),

$$\begin{aligned} (2.26) \quad k \int_0^{2\pi} G(|w|) dt &\leq A(w) - k \int_0^{2\pi} \mathcal{L}(w) dt \\ &= \|w\|_{L^2}^2 (A(\zeta) - k \int_0^{2\pi} \mathcal{L}(\zeta) dt) \\ &\leq \beta^2 \|w\|_{L^2}^2 (A(e_k) - k \int_0^{2\pi} \mathcal{L}(e_k) dt) \leq \frac{1}{a_3} \|w\|_{L^2}^2 \frac{1}{\min_j \mu_j} \end{aligned}$$

By (2.20), letting  $\Lambda_\sigma = \{\tau \in [0, 2\pi] \mid |w(\tau)| \geq 2\sigma\}$ , we have

$$\begin{aligned}
 (2.27) \quad \int_0^{2\pi} G(|w|) d\tau &\geq \int_{\Lambda_\sigma} G(|w|) d\tau \geq \frac{G(2\sigma)}{(2\sigma)^{\bar{\mu}}} \int_{\Lambda_\sigma} |w|^{\bar{\mu}} d\tau \\
 &\geq \frac{G(2\sigma)}{(2\sigma)^{\bar{\mu}}} \|w\|_{L^{\bar{\mu}}}^{\bar{\mu}} - 2\pi G(2\sigma) \geq \\
 &\geq a_4 \frac{G(2\sigma)}{(2\sigma)^{\bar{\mu}}} \|w\|_{L^2}^{\bar{\mu}} - 2\pi G(2\sigma)
 \end{aligned}$$

Combining (2.26) and (2.27) leads to

$$(2.28) \quad a_5 \|w\|_{L^2}^2 \geq k \left[ a_4 \frac{G(2\sigma)}{(2\sigma)^{\bar{\mu}}} \|w\|_{L^2}^{\bar{\mu}} - 2\pi G(2\sigma) \right]$$

which implies

$$\begin{aligned}
 (2.29) \quad \|w\|_{L^2} &\leq \max \left[ \left( \frac{4(2\sigma)^{\bar{\mu}}}{a_4} \right)^{\frac{1}{\bar{\mu}}}, \left( \frac{2a_5(2\sigma)^{\bar{\mu}}}{a_4 G(2\sigma)k} \right)^{\frac{1}{\bar{\mu}-2}} \right] \\
 &\equiv \max(a_6 \sigma, a_7(\sigma)k)^{-\frac{1}{\bar{\mu}-2}}
 \end{aligned}$$

Therefore by (2.21), (2.24), and (2.29),

$$(2.30) \quad d_k \leq a_8 \max(a_6^2 \sigma^2, a_7(\sigma)^2 k)^{-\frac{2}{\bar{\mu}-2}}$$

so

$$(2.31) \quad \lim_{k \rightarrow \infty} d_k \leq a_8 a_6^2 \sigma^2$$

Since  $\sigma \in (0, \frac{1}{2})$  is arbitrary, it follows that  $\lim_{k \rightarrow \infty} d_k = 0$  and  $Z = 0$ .

It remains to show that  $\zeta_k \rightarrow 0$  uniformly in  $C^1(\mathbb{R}, \mathbb{R}^{2n})$ . By (2.12) for all  $s > 0$ ,

$$(2.32) \quad c_k \geq a_2 \int_0^{2\pi k} \hat{H}(t, \zeta_k) dt \geq \min_{t \in \mathbb{R}, |\zeta|=s} \hat{H}(t, \zeta) m_k(s)$$

where  $m_k(s)$  denotes the measure of  $\{t \in [0, 2\pi k] \mid |\zeta_k(\frac{t}{k})| > s\}$ . Since  $c_k \rightarrow 0$ ,  $m_k(s) \rightarrow 0$  as  $k \rightarrow \infty$ . Choose any  $\varepsilon > 0$ . Then for any large  $k$  and any  $t \in \mathbb{R}$ , there is a  $\sigma \in \mathbb{R}$  such that  $|t - \sigma| \leq m_k(\varepsilon)$  and  $|\zeta_k(\frac{\sigma}{k})| \geq \varepsilon$ . Therefore

$$(2.33) \quad \begin{aligned} |\zeta_k(\frac{t}{k})| &\leq |\zeta_k(\frac{\sigma}{k})| + \int_{\sigma}^t \left| \frac{d\zeta_k}{dx} \right| dx \leq \varepsilon + m_k(\varepsilon)M \\ &\leq \varepsilon + m_k(1)M \end{aligned}$$

where  $M$  is an upper bound for  $\|\zeta_k\|_{C^1}$ . Therefore  $\|\zeta_k\|_{L^\infty} \rightarrow 0$  as  $k \rightarrow \infty$  and (1.1) implies the same for  $\dot{\zeta}_k$ . The proof is complete.

Next we study the behavior of the minimal period of  $\zeta_k$  as  $k \rightarrow \infty$ . As was mentioned in the introduction, the minimal period need not tend to infinity. Indeed if  $H(z) = g(|z|^2)$  with  $z \in \mathbb{R}^2$  and  $g'(s) \geq 1$ , the minimal period of any solution of (1.1) cannot exceed  $\pi$ . (See Remark 2.56 of [1]). The next result however gives a criterion for periodic solutions of (1.1) which converge to 0 to have a long minimal period.

Theorem 2.34: Suppose  $H \in C^1(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$  and satisfies

(H<sub>6</sub>)  $H(t, z) = \bar{H}(t, z) + R(t, z)$  where  $\bar{H}$  is positive for  $z \neq 0$ , homogeneous of degree  $\beta > 2$  in  $z$ , and  $R(t, z) = o(|z|^\beta)$ ,  $R_z(t, z) = o(|z|^{\beta-1})$  at  $z = 0$ .

If  $z_k$  are a family of periodic solutions of (1.1) and  $z_k \rightarrow 0$  uniformly as  $k \rightarrow \infty$ , then the minimal period of  $z_k \rightarrow \infty$  as  $k \rightarrow \infty$ .

Proof: Let  $T_k$  be the minimal period of  $z_k$  and set  $\tau = 2\pi t T_k^{-1}$ . Then  $w_k(\tau) \equiv z_k(t)$  is  $2\pi$  periodic and satisfies

$$(2.35) \quad \frac{dw_k}{d\tau} = \frac{T_k}{2\pi} J_{H_z} \left( \frac{T_k \tau}{2\pi}, w_k \right).$$

Let  $[z]$  denote the mean value of a  $2\pi$  periodic function  $z(t)$ , i.e.

$$[z] = \frac{1}{2\pi} \int_0^{2\pi} z(t) dt$$

If  $w_k = (u, v)$ , then (2.35) implies

$$(2.36) \quad 2 \int_0^{2\pi} (v, \dot{u})_{\mathbb{R}^{2n}} d\tau = \frac{T_k}{2\pi} \int_0^{2\pi} (w_k, H_z \left( \frac{T_k \tau}{2\pi}, w_k \right))_{\mathbb{R}^{2n}} d\tau$$

$$\leq 2 \|v - [v]\|_{L^2} \|u\|_{L^2} \leq 2 \|\dot{w}_k\|_{L^2}^2 = 2 \left( \frac{T_k}{2\pi} \right)^2 \|H_z(\cdot, w_k)\|_{L^2}^2$$

Consequently since  $w_k \rightarrow 0$  as  $k \rightarrow \infty$ , by  $(H_6)$  and (2.36),

$$(2.37) \quad M_1 \int_0^{2\pi} |w_k|^\beta d\tau \leq \int_0^{2\pi} (w_k, H_z \left( \frac{T_k \tau}{2\pi}, w_k \right))_{\mathbb{R}^{2n}} d\tau \leq \frac{T_k}{\pi} \|H_z(\cdot, w_k)\|_{L^2}^2$$

$$\leq M_2 T_k \int_0^{2\pi} |w_k|^{2(\beta-1)} d\tau \leq M_2 T_k \|w_k\|_{L^\infty}^{\beta-2} \int_0^{2\pi} |w_k|^\beta d\tau$$

which implies

$$(2.38) \quad T_k \geq \frac{M_3}{\|w_k\|_{L^\infty}^{\beta-2}} \rightarrow \infty$$

as  $k \rightarrow \infty$ .

Note that the quadratic part  $\mathcal{Q}$  of  $H$  vanishes identically in the above theorem. When this is not the case, there is a classical result of Birkhoff and Lewis [14,15] which under appropriate conditions on  $\mathcal{Q}$  and the quartic

part of  $H$  at  $0$  guarantees the existence of a family of subharmonic solutions  $z_k$  of (1.1) which converge to  $0$  and have minimal periods tending to  $\infty$  as  $k \rightarrow \infty$ . We will next show how a partial version of this local result follows from our global framework.

Theorem 2.39: Suppose  $H \in C^1(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$  and satisfies  $(H_1)$ ,  $(H_2)$ ,  $(H_4)$ , and

$(H'_3)$  There is a  $\mu \in (2, \infty)$  and  $r > 0$  such that  $0 < \mu \hat{H}(t, z) \leq (z, \hat{H}_z(t, z))_{\mathbb{R}^{2n}}$  for  $0 < |z| < 2r$

Then (1.1) possesses a sequence of  $kT$  periodic solutions  $z_k$  which converge to  $0$  as  $k \rightarrow \infty$ .

Proof: Let  $\chi_\omega(s)$  be as in Theorem 2.11. Set

$$\tilde{H}(t, z) = \mathcal{L}(z) + \chi_{\frac{r}{2}}(|z|) \hat{H}(t, z) + (1 - \chi_{\frac{r}{2}}(|z|)) \rho |z|^4$$

Then for  $\rho$  sufficiently large  $\hat{H}$  satisfies  $(\hat{H}_3)$  with  $\mu$  replaced by  $\bar{\mu} = \min(\mu, 4)$ . Consider

$$(2.40) \quad \dot{z} = \mathcal{J} \hat{H}_z(t, z).$$

By Theorem 1.36, (2.40) possesses a sequence  $z_k$  of  $kT$  periodic solutions and by Theorem 2.11,  $z_k \rightarrow 0$  as  $k \rightarrow \infty$  uniformly in  $C^1(\mathbb{R}, \mathbb{R}^{2n})$ . Hence for large  $k$ ,  $\|z_k\|_{L^\infty} < r$  and  $z_k$  satisfies (1.1)

Remark 2.37: A similar result obtains if  $\mathcal{L} \equiv 0$  and then if  $H$  satisfies  $(H_6)$ , the minimal period of  $z_k \rightarrow \infty$  as  $k \rightarrow \infty$ .

### §3 THE SUBQUADRATIC CASE

In this final section analogues of the results of §1-2 will be obtained for subquadratic Hamiltonian systems. Suppose  $H$  satisfies  $(H_4)$  and

$(H_7)$  There is a  $\nu \in (1,2)$  and  $R > 0$  such that

$$0 < (z, H_z(t, z))_{\mathbb{R}^{2n}} \leq \nu H(t, z) \quad \text{for } |z| > R,$$

$$(H_8) \quad \lim_{|z| \rightarrow \infty} \frac{|H_z(t, z)|}{|z|} \leq \varepsilon < \frac{1}{2}$$

$(H_9)$  There are constants  $a_1, a_2 > 0$  and  $s \in (1, \nu)$  such that  $H(t, z) \geq a_1 |z|^s - a_2$  for  $t \in \mathbb{R}, z \in \mathbb{R}^{2n}$ .

It was shown in [2, Theorem 4.11] that if  $H \in C^1(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$  and satisfies the above conditions and  $E$  is as in §1, then

$$g(z) = \int_0^T [H(t, z) - (p, \dot{q})_{\mathbb{R}^{2n}}] dt$$

satisfies  $(f_1) - (f_4)$  of Lemma 1.7. Thus  $g$  has a critical point and (1.1) a corresponding  $T$  periodic solution. For the verification of  $(f_1) - (f_4)$  one now takes  $E_1 = E^0 \oplus E^-$  and  $E_2 = E^+$  where  $E^0$  is the null space of  $A$  in  $E$ , and  $E^+, E^-$  are subspaces of  $E$  on which  $A$  is respectively positive and negative definite. These three subspaces are orthogonal both in  $E$  and in  $(L^2(S^1))^{2n}$ . Indeed  $E = E^0 \oplus E^+ \oplus E^-$  and we have

$$\|z\|^2 = A(z^+) - A(z^-) + |z^0|^2$$

where  $z = z^0 + z^+ + z^- \in E$ . (See [2]) Lastly the appropriate sort of linking here is provided by choosing  $S = \{v\} + E_1$  and  $Q = B_R \cap E_2$  where  $R$  is sufficiently larger than  $\|v\|$ . Actually in [2] the choice of  $v = 0$  was made. However the proof is unaffected if any  $v \in E_2$  is chosen provided that  $R = R(\|v\|)$  is suitably large. This freedom will be crucial for our next

result which establishes the existence of subharmonics for the present situation.

Theorem 3.2: Suppose  $H \in C^1(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$  and satisfies  $(H_4), (H_7)-(H_9)$ .

Then for all  $k \in \mathbb{N}$ , (1.1) possesses a  $kT$  periodic solution and

$$\|z_k\|_{L^\infty} \rightarrow \infty \text{ as } k \rightarrow \infty.$$

Proof: As usual we set  $T = 2\pi$ . Consider

$$(3.3) \quad g_k(z) = k \int_0^{2\pi} H(k\tau, z) d\tau - A(z).$$

By Theorem 4.11 of [2] cited above,  $g_k$  possesses a critical value  $c_k$  for all  $k \in \mathbb{N}$  and a corresponding critical point satisfying (1.37). As in §1, if the functions  $z_k$  were all the same, we would have  $c_k = g_k(z_k) = k c_1$ . We will show in fact that for large  $k$ ,

$$(3.4) \quad c_k \geq a_3 k^\beta$$

for some  $a_3 > 0$  and  $\beta > 1$ . Moreover since

$$c_k = g_k(z_k) = k \int_0^{2\pi} [\hat{H}(k\tau, z_k) - \frac{1}{2} (z_k, \hat{H}_z(k\tau, z_k))_{\mathbb{R}^{2n}}] d\tau,$$

if  $\{\|z_k\|_{L^\infty}\}$  were bounded, we would have

$$c_k \leq kM_1 \|z_k\|_{L^\infty} \leq M_2 k$$

contrary to (3.4). Thus (3.4) implies  $\|z_k\|_{L^\infty} \rightarrow \infty$  as  $k \rightarrow \infty$ .

It remains to verify (3.4). This estimate hinges on making a suitable choice of  $v_k \in E_2$ . By Remark 1.9,

$$(3.5) \quad c_k \geq \inf_S g_k.$$

For  $z \in S$  we have  $z = z^0 + z^- + \delta u$   $E^0 \oplus E^- \oplus \text{span } \{u\}$  where  $u \in \partial B_1 \cap E_2$ ,

and  $\delta$  is free for the moment. Therefore by  $(H_9)$ ,

$$(3.6) \quad g_k(z) \geq k \int_0^{2\pi} (a_1 |z^0 + z^- + \delta u|^s - a_2) d\tau + \|z^-\|^2 - \delta^2$$

To estimate the integral term, arguing as in [2], let  $\sigma^{-1} = 1 - s^{-1}$ . The embedding of  $\hat{E}$  in  $(L^\sigma(S^1))^{2n}$  is continuous. Therefore  $\hat{E}$ , the negative norm dual of  $E$ , contains  $(L^s(S^1))^{2n}$  [17]. Hence

$$(3.7) \quad \|z\|_{L^s} \geq a_4 \|z\|_{\hat{E}}.$$

By definition,

$$(3.8) \quad \|z\|_{\hat{E}} = \sup_{\|w\| \leq 1} (z, w)_{L^2} = \sup_{\|w\| \leq 1} [(z^0, w^0)_{L^2} + (z^-, w^-)_{L^2} + (z^+, w^+)_{L^2}]$$

Choosing  $w = u / \|u\|$ , we find

$$(3.9) \quad g_k(z) \geq a_5 k(\delta^s - 1) + \|z^-\|^2 - \delta^2$$

Setting  $\delta = k^\gamma$  shows

$$(3.10) \quad g_k(z) \geq a_6 k(k^{s\gamma} - 1) - k^{2\gamma}.$$

Finally choose  $\gamma$  so that

$$(3.11) \quad 1 + s\gamma > 2\gamma > 1$$

which is possible via our choice of  $s$  in  $(H_9)$ . Hence (3.4) obtains with

$$\beta = 1 + s\gamma.$$

An improved version of Theorem 3.2 can be obtained if  $H$  is a simple sum of kinetic and potential energy terms. Suppose e.g.  $H(z) = \frac{1}{2} |p|^2 + V(t, q)$ . Then (1.1) reduces to a second order system

$$(3.12) \quad \ddot{q} + V_q(t, q) = 0$$

which formally is the Euler equation arising from the functional

$$g(z) = \int_0^T [V(t, q) - \frac{1}{2} |q|^2] dt$$

In this setting the analogue of Theorem 3.2 holds without assuming  $(H_8)$ .

Corollary 3.13: If  $V \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$  and satisfies  $(H_4)$ ,  $(H_7)$ , and  $(H_9)$  with  $q$  replacing  $z$ , then (3.12) possesses an unbounded sequence of solution  $(q_k)$  with  $q_k$  having period  $kT$ .

Proof: Since the proof is essentially the same as that of Theorem 3.2, we will only indicate the appropriate underlying spaces and why  $(H_8)$  can be dropped. (See also [2], Theorem 4.11) We take  $T = 2\pi$  as usual and  $E = (W^{1,2}(S^1))^n$  with norm

$$\|q\|^2 = \int_0^{2\pi} (|q|^2 + |\dot{q}|^2) dt$$

The appropriate choices for  $E_1, E_2$  are  $E_1 = \{q \in E \mid q = [q]\}$  and  $E_2 = \{q \in E \mid [q] = 0\}$ . Hypothesis  $(H_8)$  is used only to verify  $(f_3)$  for  $g$ . To see why it is unnecessary in this setting, suppose  $g(q_j) \leq M$  and  $g'(q_j) \rightarrow 0$  as  $j \rightarrow \infty$ . Then dropping subscripts and arguing as in (3.6)-(3.9) via  $(H_7)$  and  $(H_9)$ , for large  $j$  we have

$$\begin{aligned} (3.14) \quad M + \frac{1}{2} \|q\| &\geq g(q) - \frac{1}{2} g'(q)q = \int_0^{2\pi} [V(t, q) - \frac{1}{2} (q, v_q)] dt \\ &\geq (1 - \frac{\nu}{2}) \int_0^{2\pi} V(t, q) dt - M_1 \geq a_3 \int_0^{2\pi} |q|^s dt - a_4 \\ &\geq a_5 |\xi|^s - a_4 \end{aligned}$$

where  $\xi = [q]$ . Note that  $(H_7)$  implies that

$$(3.15) \quad v(t, q) \leq a_6 |q|^v + a_7$$

for all  $t \in \mathbb{R}$ ,  $q \in \mathbb{R}^n$ . Since  $g'(q_j) \rightarrow 0$  as  $j \rightarrow \infty$ , for large  $j$ ,

$$(3.16) \quad |g'(q_j)\psi| \leq \|\psi\|$$

for all  $\psi \in E$ . Setting  $\psi = q_j$  and dropping the  $j$  yields

$$(3.17) \quad \|\dot{q}\|_{L^2}^2 \leq \left| \int_0^{2\pi} (q, v_q(t, q))_{\mathbb{R}^n} dt \right| + \|q\|$$

$$\leq a_6 \|q\|_L^v + \|q\| + a_8$$

via  $(H_7)$  and (3.15). Combining (3.17) with (3.14) provides a bound for  $\|q_j\|$  and  $(f_3)$  is verified.

Remark 3.18: A variant of Corollary 3.13 was obtained recently by Clarke and Ekeland [5] who replace  $(H_7)$  by (3.15) but assume  $V$  is strictly convex. Their approach to (3.15) is rather different from ours and is based on arguments from convex analysis, in particular use of a Legendre transform to reduce the problem to that of minimizing a functional.

Our final result addresses the question of when the  $kT$  periodic solutions obtained in Theorem 3.2 have minimal periods which tend to  $\infty$  with  $k$ .

Theorem 3.19: Suppose  $H \in C^1(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$  and satisfies  $(H_7)$ ,  $(H_9)$  with  $s = v$ ,

$$(H'_8) \quad \lim_{|z| \rightarrow \infty} \frac{|H_z(t, z)|}{|z|} \rightarrow 0,$$

and

$$(H_{10}) \quad H(t, z) \text{ is strictly convex in } z.$$

Let  $z_k$  be a family of periodic solutions of (1.1) such that  $\|z_k\|_{L^\infty} \rightarrow \infty$

as  $k \rightarrow \infty$ . Then the minimal period of  $z_k \rightarrow \infty$  as  $k \rightarrow \infty$ .

Proof: Suppose the minimal period of  $z_k$  is  $T_k$ . Letting  $\tau = 2\pi t T_k^{-1}$  and  $w_k(\tau) \equiv z_k(t)$ , then as in Theorem 2.34,  $w_k(\tau)$  is  $2\pi$  periodic and satisfies (2.35). Let  $\xi_k = [w_k]$  and  $\bar{w}_k = w_k - \xi_k$ . Then (2.35) implies

$$(3.20) \quad a_3 \| \bar{w}_k \|_{L^\infty} \leq \| \dot{\bar{w}}_k \|_{L^2} \leq a_4 T_k \| H_z \left( \frac{T_k \tau}{2\pi}, w_k \right) \|_{L^2} \\ \leq a_5 T_k \| H_z \left( \frac{T_k \tau}{2\pi}, w_k \right) \|_{L^\infty}$$

By  $(H'_8)$ , since  $\| w_k \|_{L^\infty} \rightarrow \infty$  as  $k \rightarrow \infty$ , given any  $\varepsilon > 0$ ,

$$(3.21) \quad \| H_z(\cdot, w_k) \|_{L^\infty} \leq \varepsilon \| w_k \|_{L^\infty}$$

for all sufficiently large  $k$ . Therefore by (3.20)-(3.21),

$$(3.22) \quad \| \bar{w}_k \|_{L^\infty} \leq a_6 T_k \varepsilon (|\xi_k| + \| w_k \|_{L^\infty})$$

If  $\{T_k\}$  is uniformly bounded e.g. by  $M$  along some subsequence, then for  $\varepsilon < (\gamma M a_6)^{-1}$  where  $\gamma > 2$  is free for the moment, we have

$$(3.23) \quad \| \bar{w}_k \|_{L^\infty} = \frac{1}{\gamma - 1} |\xi_k|$$

The strengthening of the hypotheses of Theorem 3.2 enables us to get a linear estimate for  $|\xi_k|$  in terms of  $\| \bar{w}_k \|_{L^\infty}$  next. By  $(H_9)$  we find

$$(3.24) \quad \int_0^{2\pi} H\left(\frac{T_k \tau}{2\pi}, w_k\right) d\tau \geq 2\pi(a_1 |\xi_k|^\gamma - a_2)$$

as in (3.6)-(3.9).

The form of (2.35) implies

$$(3.25) \quad \int_0^{2\pi} (H_z(\frac{T_k \tau}{2\pi}, w_k), \varphi)_{\mathbb{R}^{2n}} d\tau = 0$$

for all  $\varphi \in \mathbb{R}^n$ . But (3.25) is just the Euler equation for the variational problem

$$(3.26) \quad \min_{\eta \in \mathbb{R}^{2n}} \int_0^{2\pi} H(\frac{T_k \tau}{2\pi}, w_k + \eta) d\tau$$

and by  $(H_{10})$ ,  $\eta = \xi_k$  is the unique solution of this minimization problem.

Therefore

$$(3.27) \quad \int_0^{2\pi} H(\frac{T_k \tau}{2\pi}, w_k) d\tau \leq \int_0^{2\pi} H(\frac{T_k \tau}{2\pi}, w_k + \eta) d\tau$$

for all  $\eta \in \mathbb{R}^{2n}$ . Choosing  $\eta = 0$  and applying  $(H_7)$  yields

$$(3.28) \quad \int_0^{2\pi} H(\frac{T_k \tau}{2\pi}, w_k) d\tau \leq \int_0^{2\pi} H(\frac{T_k \tau}{2\pi}, w_k) d\tau \leq a_7 \|w_k\|_{L^\infty} + a_8$$

Combining (3.24) and (3.28), we find

$$(3.29) \quad |\xi_k| \leq M_1 \|w_k\|_{L^\infty} + M_2$$

Since

$$(3.30) \quad \|w_k\|_{L^\infty} \leq |\xi_k| + \|w_k\|_{L^\infty},$$

by (3.29)-(3.30),

$$(3.31) \quad \|w_k\|_{L^\infty} \leq (M_1 + 1) \|w_k\|_{L^\infty} + M_2$$

which shows that  $\|w_k\|_{L^\infty} \rightarrow \infty$  as  $k \rightarrow \infty$ . Then for  $k$  sufficiently large,

$$(3.32) \quad |\xi_k| \leq M_3 \|w_k\|_{L^\infty}.$$

Returning to (3.23) then gives

$$(3.33) \quad \|w_k\|_{L^\infty} \leq \frac{M_3}{\gamma-1} \|w_k\|_{L^\infty}$$

Choosing  $\gamma$  so that  $M_3(\gamma-1)^{-1} < 1$ , we see (3.33) is impossible. Hence  $T_k \rightarrow \infty$  as  $k \rightarrow \infty$  and the proof is complete.

Remark 3.34: As was the case with Corollary 3.13, an improved version of Theorem 3.19 can be given in the setting of (3.12). However we will not carry out the details.

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